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Least energy radial sign-changing solution for the Schrödinger-Poisson system in $\mathbb{R}^3$ under an asymptotically cubic nonlinerity

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Resumen

We consider the following Schrödinger-Poisson system in the whole $\mathbb{R}^3$,

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ and the nonlinearity f is "asymptotically cubic" at infinity. This implies that the nonlocal term ϕu and the nonlinear term $f(u)$ are, in some sense, in a strict competition. We show that the system admits a least energy sign-changing and radial solution obtained by minimizing the energy functional on the so-called *nodal Nehari set*.

Palabras & frases claves: Schrödinger-Poisson system, variational methods, standing wave solutions, nodal Nehari set.

1. Introducción

A great attention has been given in the last decades to the so called Schrödinger-Poisson system, namely

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

due especially to its importance in many physical applications but also since it presents difficulties and challenges from a mathematical point of view.

It is known that the system can be reduced to the equation

$$-\Delta u + u + \lambda\phi_u u = f(u) \text{ in } \quad \mathbb{R}^3,$$

and that its solutions can be found as critical points in $H^1(\mathbb{R}^3)$ of the energy functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + u^2\right)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \int_{\mathbb{R}^3}F(u)dx, \tag{2}$$

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where

$$F(t) := \int_0^t f(\tau)d\tau, \quad \phi_u = \frac{1}{4\pi|\cdot|} * u^2.$$

Before anything else, we observe that ϕ_u is automatically positive and univocally defined by u; hence words like "solution", "positive", "sign-changing" always refer to the unknown u of the system.

Observe that since $\phi_u u$ is $3-$homogeneous, in the sense that

$$\phi_{tu}(tu) = t^3\phi_u u, \quad t \in \mathbb{R},$$

there is a further difficulty in the problem exactly when the nonlinearity f behaves "cubically" at infinity, we say it is *asymptotically cubic*, being in this case in competition with the nonlocal term $\phi_u u$.

The number of papers which have studied the Schrödinger-Poisson system in the mathematical literature is so huge that it is almost impossible to give a satisfactory list. Indeed many papers deal with the problem in bounded domain or in the whole space (see e.g. [4,5,9,17,19,20] and the references therein) and some other papers deal with the fractional counterpart (see e.g [10,16] and its references). In all the cited papers various type of solutions have been found under different assumptions on the nonlinearity. However the solutions found are positive or with undefined sign and the nonlinearity f is "supercubic" at infinity (in a sense that will be specified below) an this fact helps in many computations since it gains on the nonlocal term $\phi_u u$.

Nevertheless some results have been obtained also in the asymptotically cubic case: for example, in the remarkable paper [3] the authors consider the existence of solutions under a very general nonlinearity f of Berestycki-Lions type. However they found a *positive* solution, for small values of the parameter $\lambda > 0$.

However beside the existence of positive solutions it is also interesting to find sign-changing solutions and indeed many authors began recently to address this issue. We cite the interesting paper [21] which deals with the case $f(u) = |u|^{p-1}u$ and $p \in (3, 5)$ and where the authors search for the *radial least energy sign-changing solution*, that is the radial sign-changing solution whose functional has minimal energy among all the others sign-changing solutions which are radial. Their idea is to study the energy functional on a new constraint, a subset of the Nehari manifold which contains all the sign-changing solutions.

Another interesting paper is [2] which deals with a more general nonlinearity, not necessarily of power type, where the authors assume that

- $\displaystyle \lim_{t\to\infty} \frac{F(t)}{t^4} = +\infty.$

In this sense [2] and [21] deal with a *supercubic* nonlinearity f.

The above condition is also required in [1], for the case of the bounded domain, and in [12], for the case of the whole space, where a least energy sign-changing solution is obtained.

In all these papers concerning sign-changing solutions, one of the main task is to prove that the new constraint on which minimize the functional is not empty. To show this, the fact that the nonlinearity is supercubic is strongly used.

Motivated by the previous discussion, a natural question which arises concerns the case when the nonlinearity is "cubic" at infinity. More specifically in this paper we address the problem (1) under the following conditions. Let $\lambda > 0$ and assume

(f1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f2) $f(t) = -f(-t)$ for $t \in \mathbb{R}$;

(f3) $\lim_{t\to 0} f(t)/t = 0$;

(f4) $\lim_{t\to\infty} f(t)/t^3 = 1$ and $f(t)/t^3 < 1$ for all $t \in \mathbb{R} \setminus \{0\}$;

(f5) the function $t \mapsto f(t)/t^3$ is strictly increasing on $(0, \infty)$;

(f6) recalling that $F(t) = \int_0^t f(\tau)d\tau$,

$$\lim_{t\to\infty} [f(t)t - 4F(t)] = +\infty.$$

Assumption (f4) is what we called *asymptotically cubic* behaviour for the nonlinearity and (f6) is the analogous of the usual *non-quadraticity condition*.

Our main result is the following

Teorema 1. *For any $\lambda > 0$, under the conditions (f1)-(f6), problem (1) has a radial least energy sign-changing solution. Moreover it changes sign exactly once in $\mathbb{R}^3$.*

A function f satisfying our assumptions is

$$f(t) = \frac{t^5}{1+t^2}, \quad t \in \mathbb{R},$$

which has as primitive

$$F(t) = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\ln(1+t^2).$$

Clearly this function does not satisfy the assumption

$$\lim_{t\to\infty} \frac{F(t)}{t^4} = +\infty$$

required in [2]. Moreover, the case $f(t) = |t|^{p-1}t, p \in (3, 5)$ studied in [21] does not satisfies (f4). So the present paper gives a new contribution in studying radial sign-changing solutions for the Schrödinger-Poisson problem in the asymptotically cubic nonlinearity and can be seen as a counterpart of the papers [2, 3, 21].

As we said before, we use variational methods: the solution will be found as the minimum of I, in the context of radial functions, on the constraint already introduced in [21]. Nevertheless, the main difficulty is to show that the constraint on which minimize the functional is nonempty under our assumptions on f; indeed all the techniques of the above cited papers concerning the supercubic case (see also [14] for the single equation) do not work and some new ideas have been necessary. However as a general strategy to attack the problem we follows the steps in [14].

Finally, we would like to quote the interesting paper [8] where the authors consider the asymptotically cubic (and also the super-cubic) case by assuming different conditions then ours. The authors prove the existence of a radial ground state sign-changing solution for any value of $\lambda > 0$. However their assumptions are different from ours and even the fact that the constraint where minimize is nonempty is proved in a very different way: indeed we prove this fact by making use of a suitable positive radial function $\mathfrak{u}$. For this reason, our work can be seen also as complementary to [8].

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